Let P^k (resp P_+^k) $= \{\lambda \in P$ (resp P_+) $\mid \langle \lambda, K \rangle = k\}$

§12.5 $\Rightarrow$ Weight system $P(\Lambda)$ of $L(\Lambda)$ $(\Lambda \in P_+)$ over an affine algebra $g(A)$

Prop. 12.5.

告诉我们 $\Lambda \in P_+$ 时. $\forall \mu \in P(V) \Leftrightarrow \exists w \in W$ s.t $w(\mu) \in P_+$ and $w(\mu) \le \Lambda$

(后面 maxi weight 也可以由 $P_+ \cap \max(\Lambda)$ w 作用得到)

Proposition 12.5. *Let $L(\Lambda)$ be an integrable module of positive level k over an affine algebra. Then* $\rightarrow \Lambda \in P_+$

a) *$P(\Lambda) = W \cdot \{\lambda \in P_+ | \lambda \le \Lambda\}$.*

b) *$P(\Lambda) = (\Lambda + Q) \cap$ convex hull of $W \cdot \Lambda$;*

c) *If $\lambda, \mu \in P(\Lambda)$ and μ lies in the convex hull of $W \cdot \lambda$, then $\mathrm{mult}_{L(\Lambda)} \mu \ge \mathrm{mult}_{L(\Lambda)} \lambda$.*

d) *$P(\Lambda)$ lies in the paraboloid $\{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \,|\, |\overline{\lambda}|^2 + 2k(\lambda|\Lambda_0) \le |\Lambda|^2;\ \langle \lambda, K\rangle = k\}$; the intersection of $P(\Lambda)$ with the boundary of this paraboloid is $W \cdot \Lambda$.*

e) *For $\lambda \in P(\Lambda)$ the set of $t \in \mathbb{Z}$ such that $\lambda - t\delta \in P(\Lambda)$, is an interval $[-p, +\infty)$ with $p \ge 0$, and $t \mapsto \mathrm{mult}_{L(\Lambda)}(\lambda - t\delta)$ is a nondecreasing function on this interval. Moreover, if $x \in \mathfrak{g}_{-\delta}$, $x \ne 0$, then the map $x : L(\Lambda)_{\lambda - t\delta} \to L(\Lambda)_{\lambda-(t+1)\delta}$ is injective.*

f) *Set $\mathfrak{n}_-^{(\delta)} = \bigoplus_{n>0} \mathfrak{g}_{-n\delta}$; then $L(\Lambda)$ is a free $U(\mathfrak{n}_-^{(\delta)})$-module.*

Proof. a) follows from Proposition 11.2 b), while b) and c) are special cases of Proposition 11.3 a) and b). d) follows from Proposition 11.4 a) and formula (6.2.7). e) follows from Corollary 11.9 b). f) is a special case of Proposition 11.9 c). □

Pf (a) follow from prop 11.2 (b)

[$\{i \mid \langle \Lambda, \alpha_i^\vee \rangle = 0\} \subset S(A)$ is a disjoint union of diagram of finite type $\Rightarrow$

$P(\Lambda) = W \cdot \{\lambda \in P_+ \mid \lambda \le \Lambda\}$]

Since level $= k = \langle \Lambda, K \rangle > 0$ $\exists i$ s.t $\Lambda(\alpha_i) > 0$

i.e $\{i \mid \Lambda(\alpha_i^\vee) = 0\} \subset S(A)$ is a disjoint union of diagram... ✓

(b) (c) are special case of prop 11.3 (a) and b

[prop 11.3 (a), (b) : $(\Lambda \in P_+)$ $P(\Lambda) = (\Lambda + Q) \cap$ Convex hull $(W \cdot \Lambda)$

$\hookrightarrow$ If $\lambda, \mu \in \mathfrak{h}^*$ s.t $\lambda - \mu \in Q$ $\mu \in$ Convex hull $(W \cdot \lambda)$

then $\mathrm{mult}_{L(\Lambda)}(\mu) \ge \mathrm{mult}_{L(\Lambda)}(\lambda)$

在 prop 11.3 (a) $L(\Lambda)$ over a Kac-Moody algebra $g(A)$

here $\rightarrow$ special case (i.e affine type) ✓

(d) follow from prop 11.4 a), and formula (6.2.7)

[$\Lambda \in P_+$, $\lambda, \mu \in P(\Lambda)$ then $(\Lambda|\Lambda) - (\lambda|\mu) \ge 0$ and "$=$" $\Leftrightarrow$ $\lambda = \mu \in W \cdot \Lambda$

(6.2.7): $\lambda = \overline{\lambda} + \langle \lambda, K \rangle \Lambda_0 + (\lambda|\Lambda_0)\delta$

$\langle \lambda, K \rangle = k$

paraboloid → 抛物面 $\Rightarrow$ $P(\Lambda) \subset \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid |\overline{\lambda}|^2 + 2k(\lambda|\Lambda_0) \le |\Lambda|^2\}$

and $P(\Lambda) \cap \{\lambda \in \mathfrak{h}^* \mid |\overline{\lambda}|^2 + 2k(\lambda|\Lambda_0) = |\Lambda|^2\} = W \cdot \Lambda$ ✓

$\forall\ \lambda \in P(\Lambda)$ $\quad (\Lambda|\Lambda) - (\lambda|\lambda) \geq 0 \quad (\lambda|\lambda) \leq (\Lambda|\Lambda)$

取特殊的

$(\lambda|\lambda) = |\bar{\lambda} + \langle\lambda,K\rangle\Lambda_0 + (\lambda|\Lambda_0)\delta|^2 = (\bar{\lambda}|\bar{\lambda}) + (\bar{\lambda}|\langle\lambda,K\rangle\Lambda_0) + (\bar{\lambda}|\cdots\delta) +$

但 $\lambda \in W\cdot\Lambda$ $\quad \langle\lambda,K\rangle(\Lambda_0|\bar{\lambda}) + (\Lambda_0|\delta)(\lambda|\Lambda_0) + \cdots(\Lambda_0|\Lambda_0) +$

$= |\bar{\lambda}|^2 + 2(\lambda|\Lambda_0)\langle\lambda,K\rangle \leq |\Lambda|^2$ $\quad + \langle\lambda,K\rangle(\lambda_0|\delta)$

问题，这里是 $\langle\lambda,K\rangle = \langle\Lambda,K\rangle = k$.

$(\bar{\lambda}|\Lambda_0)$ 还是只是记作 k 而已？ $\quad \nu(K) = \delta$, $(\alpha_i^\vee|d) = 0$, $\nu(d) = a_0\Lambda_0$, $(d|d) = 0$, $(\alpha_0^\vee, d) = a_0$

$(\bar{\lambda}|\delta)$ $\quad \{\alpha_0, \alpha_1, \cdots, \alpha_l, \Lambda_0\}$ is the basis of $\mathfrak{h}^*$ $\quad \{\alpha_0^\vee, \cdots, \alpha_l^\vee, d\} \to$ basis of $\mathfrak{h}$

$$\begin{cases} (\delta|\alpha_i) = 0 \quad i = \overline{0, l} \\ (\Lambda_0|\Lambda_0) = 0 \\ (\alpha_i|\Lambda_0) = 0 \end{cases} \Longrightarrow \quad (\delta|\Lambda_0) = 1 \quad (\bar{\lambda}|\Lambda_0) \quad (\alpha_0|\Lambda_0) = \frac{1}{a_0} \quad \alpha_0 = \frac{1}{a_0}(\delta - \theta)$$

(e) follow from Coro 11.9. (b)

[Let $\alpha \in \Delta_+^{im}$ and $\Lambda \in P_+$, Let $\lambda \in P(\Lambda)$, $(\lambda|\alpha) \neq 0$, then $(\lambda|\alpha) > 0$

(虚根的权链) then $\Longrightarrow \begin{cases} \{\lambda \mid \lambda - t\alpha \in P(\Lambda)\} = \{t \mid t \in [-p, +\infty),\ p \geq 0\} \\ t \longmapsto \mathrm{mult}_{L(\Lambda)}(\lambda - t\alpha) \text{ is a nondecreasing function on} \end{cases}$

$\lambda \in P(\Lambda)$

$(\lambda|\delta) = 0 \quad \lambda - t\alpha \in P(\Lambda) \Rightarrow t = 0 \quad t \in [-p, +\infty)$, $p \geq 0$

$(\lambda|\delta) \neq 0 \quad [\ (\lambda|\alpha) \geq 0$ if $\lambda \in P(\Lambda)$. $\alpha \in \Delta_+^{im}$]

$\Rightarrow (\lambda|\delta) > 0$

(f) is special case of prop 11.9 (c)

[$L(\Lambda)_+^{(\alpha)}$ is free $U(n_-^{(\alpha)})$-module on a basis of the subspace

$\{x \in L(\Lambda)_+^{(\alpha)} \mid n_+^{(\alpha)}(x) = 0\}$

$\qquad \| $

$\bigoplus_{\lambda: (\lambda|\alpha) > 0} L(\Lambda)_\lambda$]

$$L(\Lambda) = L(\Lambda)_0^{(\delta)} \oplus L(\Lambda)_+^{(\delta)}$$, where $\underline{g^{(\delta)} \cdot L(\Lambda)_0^{(\delta)} = 0}$

$$\begin{aligned} &\quad\ \ \| \\ &\bigoplus_{\lambda : (\lambda|\delta)=0} L(\Lambda)_\lambda \oplus \Big(\bigoplus_{\lambda : (\lambda|\delta)>0} L(\Lambda)_\lambda \Big) \end{aligned}$$

i.e $L(\Lambda)_+^{(\delta)}$ is free $U(n_-^{(\delta)})$-module

$\Downarrow$

$L(\Lambda)$ is free $U(n_-^{(\delta)})$-module. ✓

§12.6 (Weight system $P(\Lambda)$ ($\Lambda \in P_+$) of positive level k over an affine alg)

Def: A weight $\lambda \in P(\Lambda)$ is called maximal if $\lambda + \delta \notin P(\Lambda)$, denote by $\max(\Lambda)$ the set of all maximal weight of $L(\Lambda)$

Fact 1: $\max(\Lambda)$ is a W-invariant set

[$w(\delta) = \delta$ $\quad w(\lambda+\delta) = w(\lambda) + \delta \in P(\Lambda) \Rightarrow w(\lambda) \in \max(\lambda)$

if $w(\lambda+\delta) \in P(\Lambda) \Rightarrow \lambda+\delta \in P(\Lambda)$ & $\lambda+\delta \notin P(\Lambda)$]

Fact 2: $\forall\, u \in \max(\Lambda)$, u is W-equivalent to a unique dominat weight.

[By prop 12.5. $\quad P(\Lambda) = W\{\lambda \in P_+ \mid \lambda \le \Lambda\}$

$\left.\begin{array}{l} \exists\ \lambda \in P_+ \\ \exists\ w \in W \end{array}\right\}$ s.t $\quad w(\lambda) = u$

Since $\lambda = w(u) \in \max(\Lambda)$ by Fact 1 $\Rightarrow \lambda \in P_+ \cap \max(\Lambda)$

(unique.) $\quad \lambda \in P_+$, $\underline{\underline{w(u)}} \cap \underline{\underline{P_+}}$ is exactly one element by prop 3.12]

Fact 3: $\forall\, u \in P(\Lambda)$ $\ \exists\ \lambda \in \max(\Lambda)$ s.t

$u = \lambda - n\delta$, where n a unique nonnegative integer

i.e we have $P(\Lambda) = \bigcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta \mid n \in \mathbb{Z}_+\}$

(disjoint union)

[by prop 12.5 (e) + Fact 2] $\qquad$ Prop 12.5 $\quad P(\Lambda) \supseteq$

Fact 2: $\quad \boxed{P(\Lambda) \subseteq}$

prop12.6 (description of dominant maximal weights)

PROPOSITION 12.6. *The map $\lambda \mapsto \bar{\lambda}$ defines a* ~~*bijection*~~ injective *from* $\max(\Lambda) \cap P_+$ *onto* $kC_{af} \cap (\bar{\Lambda} + \bar{Q})$. *In particular, the set of dominant maximal weights of* $L(\Lambda)$ *is finite.*

Proof. Straightforward using Proposition 12.5.

Pf:

C_{af} → fundamental alcove in §6.6. → 基本付房

$C^\vee$ 里挖去一些

Recall $C_{af} = \{ \lambda \in \mathring{\mathfrak{h}}_{\mathbb{R}}^* \mid (\lambda|\alpha_i) \geq 0$ for $i = \overline{1,l}$ and $(\lambda|\theta) \leq 1$

$\theta = \sum_{i=1}^{l} a_i \alpha_i \}$

Thus: $k C_{af} = \{ \lambda \in \mathring{\mathfrak{h}}_{\mathbb{R}}^* \mid (\lambda|\alpha_i) \geq 0$ for $i = \overline{1,l}$, $(\lambda|\theta) \leq k \}$

For $u \in \max(\Lambda) \cap P_+$, we have:

$u(\alpha_i) \geq 0$ $\Rightarrow$ 推 $(u|\theta) \leq k$ for $i = \overline{0,l}$ and $u(K) = k > 0$

Since $\alpha_0^\vee = K - a_0\theta^\vee$ $(a_0^\vee = 1)$

We have $0 \leq u(\alpha_0^\vee) = \langle u, K - a_0\theta^\vee \rangle = u(K) - \langle u, a_0\theta^\vee \rangle$

$= k - \langle u, \nu^{-1}(\theta) \rangle = k - (u|\theta) = k - (\bar{u}|\theta)$

$\Rightarrow (\bar{u}|\theta) \leq k$ i.e $\bar{u} \in kC_{af}$

$\Rightarrow \bar{u} \in kC_{af} \cap \bar{\Lambda} + \bar{Q}$

(injective)

if $u_1, u_2 \in \max(\Lambda) \cap P_+$ and $\bar{u}_1 = \bar{u}_2$

then $u_1 - u_2 = (\bar{u}_1 + (u_1|\Lambda_0)\delta + \langle u_1, K \rangle \Lambda_0) - (\bar{u}_2 + \cdots)$

$= (u_1 - u_2|\Lambda_0)\delta + \langle u_1 - u_2, K \rangle \Lambda_0 = 0$

$= (u_1 - u_2|\Lambda_0)\delta$

On the other hand $u_1, u_2 \in \Lambda - Q_+$

$\Rightarrow u_1 - u_2 \in Q \Rightarrow u_1 - u_2 = m\delta$, $m \in \mathbb{Z}$

$(K|K) = 0$
$(K|\alpha_i^\vee) = 0$
$(K|d) = 0$, $d \to \Lambda_0$
$K \to \delta$ $(\delta|\delta) = 0$

But, u_1, u_2 are maximal $\Rightarrow$ $m=0$ $\Rightarrow u_1 = u_2$

(Surjective) ✓

In particular, $K C_{af} \cap \bar{\Lambda} + \bar{Q}$ is a finite set

$\Rightarrow$ $\max(\lambda) \cap P_+$ is finite set ✓

[$K C_{af}$ is a compact region
and $\bar{\Lambda} + \bar{Q}$ is a discrete set]

Prop 12.6.

Let A be an affine matrix of type $X_N^{(r)}$, where $X = A, D$ or E. ($\Lambda \in P_+$) be of level 1, Then

(12.6.2) $\max(\Lambda) = W \cdot \Lambda = T \cdot \Lambda$

(12.6.3) $P(\Lambda) = \{\Lambda_0 + \frac{1}{2}|\Lambda|^2 \delta + \alpha - (\frac{1}{2}|\alpha|^2 + s)\delta$
where $\alpha \in \bar{\Lambda} + \bar{Q}$, $s \in \mathbb{Z}_+\}$

Pf: If $w(\Lambda) + \delta \in P(\Lambda)$ for some $w \in W$
then $\Lambda + \delta \in P(\Lambda)$ $\Rightarrow$ $W(\Lambda) \subset \max(\Lambda)$

then, it suffices to prove that $\max(\Lambda) \subseteq T(\Lambda)$ 显然

$(\underbrace{W(\Lambda) \subset}_{显然} \max(\Lambda) \subseteq T(\Lambda) \underbrace{\subseteq}_{显然} W(\Lambda))$

$T = \{t_\alpha \mid \alpha \in M = \mathring{Q} = \bar{Q}\}$

Since $\text{Level}(\Lambda) = 1$, $\Lambda = \Lambda_i + c\delta$

If $i=0$ $\mathring{w}(\Lambda) = \Lambda$ $\Rightarrow$ $W(\Lambda) = T(\Lambda)$

$r_0(\Lambda_0 + c\delta) = \Lambda_0$

$i \neq 0$ $W(\Lambda_i) = T(\Lambda_i)$ $\Rightarrow$ $W(\Lambda) = T(\Lambda)$

$\forall\ \lambda \in \max(\Lambda)$, we have $\lambda = \Lambda - \beta$, wher $\beta \in Q_+$

$\beta \in Q_+$ $\beta \in Q_+$

(问题 $(\lambda | k) = \Lambda(k) = 1$ 吗?) $\lambda = \Lambda - \beta = \Lambda_i + (\epsilon\delta - \beta)$

$\langle \beta, k \rangle = 0$?

Since $\beta \equiv \bar{\beta} \mod (\mathbb{C}\delta)$, we have: $t_{\bar{\beta}}(\lambda) = \Lambda \mod \mathbb{C}\delta$ by (6.5.2)

$(i \neq 0)$

$\beta = \bar{\beta} + (\beta | \Lambda_0)\delta + \langle \beta, k \rangle \Lambda_0 \Rightarrow \beta = \bar{\beta} \mod \mathbb{C}\delta$.

$t_{\bar{\beta}}(\lambda) = \lambda + \langle \lambda, k \rangle \beta - (\lambda | \beta) + \frac{1}{2}|\beta|^2 \langle \beta, k \rangle)\delta \mod \mathbb{C}\delta$

$= \lambda + \langle \lambda, k \rangle \bar{\beta} \mod \mathbb{C}\delta$

$= \Lambda - \bar{\beta} + \underset{1}{\underset{\parallel}{\langle \lambda, k \rangle}}\bar{\beta} = \Lambda \mod \mathbb{C}\delta$

$\Rightarrow \Lambda - t_{\bar{\beta}}(\lambda) = m\delta \quad m \in \mathbb{Z}_+$

But $t_\beta(\lambda) \in \max(\Lambda) \Rightarrow m = 0 \quad t_{\bar{\beta}}(\lambda) = \Lambda \quad \lambda = t_{\bar{\beta}}(\Lambda)$

i.e. $\max(\Lambda) \subset T(\Lambda) \subset W(\Lambda) \Rightarrow (12.6.2)$ ✓

$W(\lambda) \subset$

(12.6.3) follow from (12.6.1) + (12.6.2)

$P(\Lambda) = \bigcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta \mid n \in \mathbb{Z}_+\}$

(12.6.2) $\lambda \in \max(\Lambda) = T(\Lambda)$

For $\alpha \in M \in \bar{Q} = \mathring{Q}$, we have

$t_\alpha(\Lambda) = \Lambda + \alpha - ((\Lambda | \alpha) + \frac{1}{2}(\alpha | \alpha))\delta$

$\in \mathbb{Z}_+$

(Since $\Lambda = \bar{\Lambda} + \underset{1}{\underset{\parallel}{\langle \Lambda, k \rangle}} \Lambda_0 + \frac{1}{2}(|\Lambda|^2 - |\bar{\Lambda}|^2)\delta$)

$\in \mathbb{Z}_+$

6.5.2 - 6.5.6

$= \bar{\Lambda} + \frac{1}{2}(|\Lambda|^2 - |\bar{\Lambda}|^2)\delta + \Lambda_0 + \alpha - ((\bar{\Lambda} | \alpha) + \frac{1}{2}(\alpha|\alpha))\delta$

$= \Lambda_0 + \frac{1}{2}|\Lambda|^2\delta + \bar{\Lambda} + \alpha - \frac{|\bar{\Lambda} + \alpha|^2}{2}\delta$ (let $\bar{\Lambda} + \alpha = \beta$, $\beta \in \bar{\Lambda} + \bar{Q}$)

$= \Lambda_0 + \frac{1}{2}|\Lambda|^2\delta + \beta - \frac{|\beta|^2}{2}\delta$

$P(\Lambda) = \bigcup_{\lambda \in \max(\Lambda)} \{\lambda - n\delta \mid n \in \mathbb{Z}_+\} = \{\Lambda_0 + \frac{1}{2}|\Lambda|^2\delta + \beta - \frac{|\beta|^2}{2} - n\delta\}$

$= \{\lambda_0 + \frac{1}{2}|\lambda|^2\delta + \beta - (\frac{1}{2}|\beta|^2 - s\delta) \quad s \in \mathbb{Z}_+\}$ ✓

§12.7 . $\forall\, Y = \{h \in \mathfrak{h} \mid \sum_{\alpha \in \Delta_+} \mathrm{mult}\,\alpha\, |e^{-\alpha(h)}| < \infty\}$ If A is affine then $\Rightarrow$

Let $\Lambda \in P_+$, If follow from prop 11.10 (11.10.1) $\rightarrow$

$\{Y = \{h \in \mathfrak{h} \mid \mathrm{Re}\langle\delta, h\rangle > 0\}$, $ch_{L(\Lambda)}$ converges absolutely to a holomorphic fuction in $Y = \{h \in \mathfrak{h} \mid \mathrm{Re}\,\delta(h) > 0\}$

In Fact, Y is region of convergence of $ch_{L(\Lambda)}$ if level $(\Lambda) > 0$
$\Lambda(k) > 0$

Note also $\forall$ h.w.m V over an affine algebra.
ch_V converges absolutely in domain Y_0 (see Lem 10.6 b)
[then $Y(V) \supset Y \cap Y_0$]

$Y_0 = \{h \in \mathfrak{h} \mid \mathrm{Re}\langle\alpha_i, h\rangle > 0\}$ 此时 $Y \cap Y_0 = Y_0 \subset Y(V)$

$Y = \{h \in \mathfrak{h} \mid \mathrm{Re}\langle\delta, h\rangle > 0\} = Y$ 上面的 $(+)$

Def. Suppose $\Lambda \in P_+$, $\lambda \in \max(\Lambda)$, we define

$$a_\lambda^\Lambda = \sum_{n=0}^{\infty} \mathrm{mult}_{L(\Lambda)}(\lambda - n\delta)\, e^{-n\delta}$$

? $(e^{-n\delta}\underline{\underline{e(h)}})$

a_λ^Λ converges absolutely in the region Y since

$$\Big[\quad ch_{L(\Lambda)} = \sum_{\lambda \in P(\Lambda)} \mathrm{mult}(\lambda)\, e^{\lambda}$$
$$= \sum_{\lambda \in \max(\Lambda)} \boxed{\sum_{n=0}^{\infty} \mathrm{mult}_{L(\Lambda)}(\lambda - n\delta)\, e^{\lambda - n\delta}}$$

$$= \sum_{\lambda \in \max(\Lambda)} e^{\lambda} \hat{a}_{\lambda}$$

Since $W_\lambda \cap T = 1$ for $\lambda \in P(\Lambda)$ (prop 6.6c) and $w(\delta) = \delta$

and using (12.6.1) $\max \Lambda = W \cdot \Lambda = T(\Lambda)$:

(12.7.1)

$$\mathrm{ch}_{L(\Lambda)} = \sum_{\lambda \in \max(\Lambda)} e^{\lambda} \hat{a}_{\lambda} = \boxed{\sum_{\substack{\lambda \in \max \Lambda \\ \Lambda \bmod T}} \left(\sum_{t \in T} e^{t(\lambda)} \right) \hat{a}_{\lambda}}$$

(Computation of $\mathrm{ch}_{L(\Lambda)}$ $\longrightarrow$ computation of the function $\hat{a}_{\lambda}$)